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Banach Algebras, Logarithms, and Polynomials of Convolution Type

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Using elementary Banach algebra techniques, it is determined which elements of Banach algebras like $l_1(\alpha)$, $\mathscr{C}(\mathscr{K})$, and $\mathscr{A}$ have logarithms. The Wiener–Lévy theorem will be used to answer the same question for more complicated Banach algebras like the Wiener algebra. These results will be applied to polynomials of convolution type and a generalization thereof.

INTRODUCTION

Existence of logarithms of functions is needed in several parts of mathematics. For example, in the theory of entire functions of a complex variable one needs that if f is a non-vanishing entire function, then there exists a entire function g such that $f(z)=e^{g(z)}$ for all $z \in \mathbb{C}$. In probability theory the following analogous result is essential for the theory of infinitely divisible probability measures: if f is a non-vanishing complex-valued continuous function on $\mathbb{R}$, then there exists a continuous function g such that $f(x)=e^{g(x)}$ for all $x \in \mathbb{R}$.

In this paper a unified approach is presented to these and related results. The approach uses only elementary Banach algebra techniques and is presented in Section 1. Although the approach is simple, it seems to be new. Sections 2 and 3 contain applications of the results of Section 1. For example, the two results mentioned above are Theorems 2.12 and 2.7. Section 4 contains applications to polynomials of convolution type. A sequence $(q_n)_{n \in \mathbb{N}}$ of polynomials is said to be of *convolution type* if degree $(q_n)=n$ for $n=0, 1, \ldots$ and

$$q_n(x+y)=\sum_{k=0}^{n} q_k(x)\, q_{n-k}(y) \tag{1}$$

holds for $n=0, 1, \ldots$ and all $x, y \in \mathbb{R}$.

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The most general solution of Eqs. (1) with arbitrary functions q_n is given in [18] (see also [1, pp. 111–116]).

If $(q_n)_{n \in \mathbb{N}}$ is a sequence of polynomials of convolution type, then $(n! q_n)_{n \in \mathbb{N}}$ is a sequence of polynomials of binomial type, as introduced in [20, 22]. Many special polynomials (e.g., Hermite, Laguerre) are of binomial type or closely related to polynomials of binomial type [22, 23]. Moreover, there are connections with combinatorics [5, 20, 21], probability theory [10, 26, 27] and approximation theory [17, 19]. The author has chosen to work with polynomials of convolution type instead of polynomials of binomial type, because convolution is a fundamental operation in Fourier analysis and probability theory.

The theory of polynomials of binomial type as developed in [20, 22] is mainly an algebraic one. We apply the results of Sections 2 and 3 to obtain new analytical results on polynomials of convolution type. The reader should consult [14] for other applications of Banach algebras to polynomials of binomial type.

Finally, in Section 5 a two-sided analogue of polynomials of convolution type is introduced and studied.

1. General Banach Algebra Techniques

The purpose of this section is to prove Theorem 1.3, which is basic to our use of Banach algebra theory. For the basics of Banach algebra theory the reader is referred to [25, Chap. 18]. In particular, [25, Th. 18.17(c)] will be used without mentioning.

We begin with fixing some notation and recalling definitions.

Notation. Let $\mathscr{B}$ be a Banach algebra with unit element e.

(1) $\operatorname{inv}(\mathscr{B}) := \{x \in \mathscr{B} \mid x^{-1} \in \mathscr{B}\}$

(2) $\exp(\mathscr{B}) := \{x \in \mathscr{B} \mid \exists y \in \mathscr{B}$ such that $x = e^y\}$. The element $e^y \in \mathscr{B}$ is defined by the power series $\sum_{n=0}^{\infty} (y^n/n!)$ which converges in $\mathscr{B}$ for all $y \in \mathscr{B}$; y^0 is defined to be the unit element of $\mathscr{B}$.

(3) $\mathscr{G}_1 :=$ component of $\operatorname{inv}(\mathscr{B})$ that contains the unit element of $\mathscr{B}$. The definition of $\mathscr{G}_1$ is with respect to the norm topology which $\operatorname{inv}(\mathscr{B})$ inherits from $\mathscr{B}$.

(4) $\Delta(\mathscr{B}) :=$ the *maximal ideal space* of $\mathscr{B}$, i.e., the set of all complex homomorphisms of $\mathscr{B}$.

Remark 1.1. If $\mathscr{B}$ is a Banach algebra $\mathscr{B}$ with unit element e, then $\exp(\mathscr{B}) \subset \operatorname{inv}(\mathscr{B})$ since $e^y e^{-y} = e$ for all $y \in \mathscr{B}$ [4, p. 49, Lemma 1.4.1].

Definition. Let T be a topological space. A subset U of T is *connected* if U is not the union of two disjoint relatively open subsets of U.

A *component of* U is a connected subset of U which is not contained in a larger connected subset of U.

It is easy to prove that components are relatively closed, that components of open sets are open, and that continuous images of connected sets are connected. Note that the union of two non-disjoint connected sets is again connected (see, e.g., [8, p. 22, Th. 1.1].

We are now ready for the first theorem which relates $\exp(\mathscr{B})$ to the topological structure of $\operatorname{inv}(\mathscr{B})$.

THEOREM 1.2. *Let $\mathscr{B}$ be a commutative Banach algebra with unit e. Then* $\exp(\mathscr{B})=\mathscr{G}_1$. *In particular,* $\exp(\mathscr{B})$ *is closed in* $\operatorname{inv}(\mathscr{B})$.

Proof. A completely elementary proof of the first statement can be found in [4, pp. 49–50]. The second statement follows from the first statement, since components are always relatively closed. ∎

Theorem 1.2 is difficult to use since there is no general method for calculating $\mathscr{G}_1$. For the algebras that will be discussed in Section 2, this problem will be solved by using the following theorem.

DEFINITION. Let T be a topological space and $a, b \in T$. A *path from a to b in T* is a continuous function $f\colon [0, 1] \to T$ such that $f(0)=a$ and $f(1)=b$.

THEOREM 1.3. *Let $\mathscr{B}$ be a commutative Banach algebra with unit e. Then* $x \in \exp(\mathscr{B})$ *if and only if* $x \in \operatorname{inv}(\mathscr{B})$ *and there is path f in* $\operatorname{inv}(\mathscr{B})$ *from αe to x for some* $\alpha \in \mathbb{C}\setminus\{0\}$.

Proof. If $x \in \exp(\mathscr{B})$, then by definition there exists $y \in \mathscr{B}$ such that $x=e^y$. Then F, defined by $F(t):=e^{ty}$, is a path from e to $e^y=x$. Conversely, let g be an arbitrary path in $\mathbb{C}\setminus\{0\}$ from 1 to α. Then h, defined by $h(t):=g(t)e$, is a path in $\operatorname{inv}(\mathscr{B})$ from e to αe. Therefore F, defined by $F(t):=h(2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $F(t):=f(2t-1)$ for $\frac{1}{2} \leqslant t \leqslant 1$, is a path in $\operatorname{inv}(\mathscr{B})$ from e to x. It follows from the continuity of F that $F([0, 1])$ is connected. Hence $F([0, 1]) \cup \mathscr{G}_1$ is connected, since $e \in F([0, 1]) \cap \mathscr{G}_1$. Because $\mathscr{G}_1$ is the largest connected subset of $\operatorname{inv}(\mathscr{B})$ that contains e, we have $\mathscr{G}_1=\mathscr{G}_1 \cup F([0, 1])$. It follows in particular that $x=F(1) \in \mathscr{G}_1$, hence $x \in \exp(\mathscr{B})$ by Theorem 1.2. ∎

Remark 1.4. Theorem 1.3 is false when the Banach algebra is not commutative. Let $\mathscr{B}(\mathscr{H})$ denote the Banach algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$. Then $\operatorname{inv}(\mathscr{B}(\mathscr{H}))$ is pathwise connected [2, p. 51, Teil I.6, Satz I] but need not be equal to $\exp(\mathscr{B}(\mathscr{H}))$ [24, Th. 12.38].

2. Algebras on the Unit Disc

In this section Theorem 1.3 will be applied to the Banach algebras $l_1(\alpha)$, $\mathscr{C}(\mathscr{K})$, and $\mathscr{A}$.

Notation. $\mathscr{D} := \{z \in \mathbb{C} : |z| < 1\}$; $\bar{\mathscr{D}} := \{z \in \mathbb{C} : |z| \leqslant 1\}$; $\mathscr{T} := \{z \in \mathbb{C} \mid |z| = 1\}$; Log denotes the principal branch of the logarithm on $\mathbb{C} \setminus (-\infty, 0]$.

For later use we begin by proving the following uniqueness lemma.

Lemma 2.1. *Let $\mathscr{K}$ be a connected subset of $\mathbb{C}$. Suppose that g and h are continuous functions on $\mathscr{K}$ such that $g(a) = h(a)$ for some $a \in \mathscr{K}$ and $e^{g(z)} = e^{h(z)}$ for all $z \in \mathscr{K}$. Then $g(z) = h(z)$ for all $z \in \mathscr{K}$.*

Proof. It follows from $e^{g(z)} = e^{h(z)}$ that $g(z) = h(z) + 2\pi i k(z)$ with $k(z) \in \mathbb{Z}$. Hence k is a continuous function on $\mathscr{K}$ with $k(a) = 0$. Since the continuous image of $\mathscr{K}$ is connected, we must have $k \equiv 0$. ∎

(a) Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers satisfying $\alpha_0 = 1$ and $\alpha_{n+m} \leqslant \alpha_n \alpha_m$ for all $n, m \in \mathbb{N}$. Let $l_1(\alpha)$ be the Banach algebra of all sequences $x = (x_n)_{n \in \mathbb{N}}$ such that $\|x\|_{1,\alpha} := \sum_{n=0}^{\infty} \alpha_n |x_n| < \infty$. Addition in $l_1(\alpha)$ is defined componentwise, multiplication is defined to be convolution. The complex homomorphisms of $l_1(\alpha)$ are of the form $x \to \sum_{n=0}^{\infty} x_n z^n$ ($|z| \leqslant e^{\rho}$; $\rho := \lim_{n \to \infty} n^{-1} \log \alpha_n$, see [15, pp. 116–120, Sect. 19]). The unit element is the sequence $(1, 0, 0, \ldots)$.

If $\alpha_n = 1$ for all $n \in \mathbb{N}$, then $l_1(\alpha)$ is the usual Banach space l_1 of absolutely summable sequences.

Theorem 2.2. $\{x \in l_1(\alpha) : \sum_{n=0}^{\infty} x_n z^n \neq 0$ *for all* $|z| \leqslant e^{\rho}\} = \operatorname{inv}(l_1(\alpha)) = \exp(l_1(\alpha))$.

Proof. The first equality follows from [25, Th. 18.17].

For the second equality we only need to prove $\operatorname{inv}(l_1(\alpha)) \subset \exp(l_1(\alpha))$. Let $x \in \operatorname{inv}(l_1(\alpha))$ be arbitrary. Then $\sum_{n=0}^{\infty} x_n z^n \neq 0$ for all $|z| \leqslant e^{\rho}$, hence in particular $x_0 \neq 0$. Define f: $[0, 1] \to l_1(\alpha)$ by $f(t) := (t^n x_n)_{n \in \mathbb{N}}$. It follows from dominated convergence that $\lim_{t \to s} \|f(t) - f(s)\|_{1,\alpha} = 0$. Hence f is a path in $\operatorname{inv}(l_1(\alpha))$ from $x_0 e$ to x. Theorem 1.3 now yields $x \in \exp(l_1(\alpha))$. ∎

For an extension of Theorem 2.2 for the special case l_1, see Theorem 3.7.

(b) Let $\mathscr{K}$ be a compact subset of $\mathbb{C}$. Denote by $\mathscr{C}(\mathscr{K})$ the Banach algebra of all continuous functions on $\mathscr{K}$ with pointwise addition and multiplication (see [24, Example 11.13(a)]). The norm on $\mathscr{C}(\mathscr{K})$ is the supremum norm, denoted by $\|f\|_{\infty}$. The unit e of $\mathscr{C}(\mathscr{K})$ is the function that is identically one.

We will prove in Theorem 2.5 that if $\mathscr{K}$ is contractible to a point (see Definition 2.3 below), then an analogue of Theorem 2.2 holds for $\mathscr{C}(\mathscr{K})$.

DEFINITION 2.3. Let $\mathscr{K}$ be a subset of $\mathbb{C}$. A *contraction* of $\mathscr{K}$ to $z_0 \in \mathscr{K}$ is a continuous mapping H from $[0, 1] \times \mathscr{K} \to \mathscr{K}$ such that $H(0, z) = z_0$ and $H(1, z) = z$ for all $z \in \mathscr{K}$. If there exists a contraction of $\mathscr{K}$ to some point of $\mathscr{K}$, then $\mathscr{K}$ is said to be *contractible to a point*.

EXAMPLES 2.4. (a) Any disc $\{z \in \mathbb{C} : |z| \leqslant r\}$ is contractible to 0: use the contraction $H(t, z) = tz$.

(b) If $a, b \in \mathbb{R}$, then $[a, b]$ is contractible to a: define $H(t, x) = a + t(x - a)$.

(c) $\mathbb{C} \backslash (-\infty, 0]$ is contractible to 1: define $H(t, re^{i\varphi}) := (tr + 1 - t)\, e^{it\varphi}$.

THEOREM 2.5. *Let $\mathscr{K}$ be a compact subset of $\mathbb{C}$ such that $\mathscr{K}$ is contractible to a point of $\mathscr{K}$. Then* $\operatorname{inv}(\mathscr{C}(\mathscr{K})) = \exp(\mathscr{C}(\mathscr{K}))$. *In particular, if f is a non-vanishing continuous function on $\mathscr{K}$, then there is a continuous function g on $\mathscr{K}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathscr{K}$. Moreover, g is analytic in those points in which f is analytic. If $f(a) = 1$ for some point $a \in \mathscr{K}$, then there is unique continuous function g on $\mathscr{K}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathscr{K}$ and $g(0) = 0$.*

Proof. We only need to prove $\operatorname{inv}(\mathscr{C}(\mathscr{K})) \subset \exp(\mathscr{C}(\mathscr{K}))$. Let $f \in \operatorname{inv}(\mathscr{C}(\mathscr{K}))$ be arbitrary and let H be an arbitrary contraction of $\mathscr{K}$ to $a \in \mathscr{K}$, say. By the uniform continuity of f and H, $\lim_{t \to s} Uf(H(t, \cdot)) - f(H(s, \cdot))\|_\infty = 0$. Hence F, defined by $F(t) := f(H(t, \cdot))$, is a path in $\operatorname{inv}(\mathscr{C}(\mathscr{K}))$ from $f(a)e$ to f. Now Theorem 1.3 yields $f \in \exp(\mathscr{C}(\mathscr{K}))$.

If f is analytic at z_0, then for all z sufficiently close to z_0, $g(z) = \log f(z_0) + \operatorname{Log}\{1 + (f(z) - f(z_0)/f(z_0))\}$, where $\log f(z_0)$ denotes some number such that $\exp(\log f(z_0)) = f(z_0)$.

The last statement follows directly from Lemma 2.1. ∎

Remark 2.6. (a) The second statement of Theorem 2.5 also holds if each component of $\mathscr{K}$ is compact and contractible.

(b) Theorem 2.5 also holds if contractibility of $\mathscr{K}$ is weakened to connectedness of $\mathbb{C} \backslash \mathscr{K}$ [8, Cor. 4.33]. The author has not been able to find a simple proof of this result with the method of this paper (cf. Remark 2.10).

The so-called topologist's sine-curve (see, e.g., [28, pp. 44–45]) is an example of a compact connected subset of $\mathbb{C}$ which is not contractible but has connected complement (this example was shown to the author by professor J. van Mill). It follows from the Alexander duality theorem [28, Chap. 11] that if $\mathscr{K}$ is a compact contractible subset of $\mathbb{C}$, then both $\mathscr{K}$ and $\mathbb{C} \backslash \mathscr{K}$ are connected.

(c) Professor J. van Mill has pointed out to the author that if $\mathscr{K}$ is a compact connected subset of $\mathbb{C}$ with connected complement, then it follows from [3, p. 322, Th. 7.6] that for each $\varepsilon > 0$, there exists a compact contractible subset $\mathscr{K}_\varepsilon$ of $\mathbb{C}$ such that $\operatorname{dist}(\mathscr{K}, z) < \varepsilon$ for all $z \in \mathscr{K}_\varepsilon$. Using this result, we can easily prove the extension of Theorem 2.5 mentioned in (b).

The special case $\mathscr{K} = [a, b]$ and the following theorem are important in probability theory, see, e.g., [11, Chap. 7, especially Th. 7.6.2].

THEOREM 2.7. *Let f be a non-vanishing continuous function on $\mathbb{R}$ such that $f(0) = 1$. Then there exists a unique continuous function g on $\mathbb{R}$ such that $f(x) = e^{g(x)}$ for all $x \in \mathbb{R}$ and $g(0) = 0$.*

Proof. It follows from Theorem 2.5 and Example 2.4(b) that there exist for each $n \in \mathbb{N}$ unique continuous functions g_n such that $e^{g_n} = f$ on $[-n, n]$ and $g_n(0) = 0$. By Lemma 2.1, $g_n \equiv g_m$ on $[-n, n]$ if $m > n$. Hence the function g, defined by $g(x) := g_n(x)$ if $|x| \leqslant n$, is well defined, continuous, satisfies $f(x) = e^{g(x)}$ for all $x \in \mathbb{R}$, and $g(0) = 0$. Uniqueness follows from Lemma 2.1. ∎

If $\mathscr{K}$ is compact, but not necessarily contractible to a point, then the following theorem due to Borsuk [8, Th. 4.24] states which continuous functions on $\mathscr{K}$ have continuous logarithms. Our proof, which is new, follows from the simple observation that a homotopy in $\mathbb{C} \backslash \{0\}$ is a path in $\operatorname{inv}(\mathscr{C}(K))$.

DEFINITION 2.8. Let $\mathscr{K}$ be a subset of $\mathbb{C}$ and let $f, g \colon \mathscr{K} \to V \subset \mathbb{C}$ be continuous functions. A *homotopy in V of f with g* is a continuous function $H \colon [0, 1] \times \mathscr{K} \to V$ such that $H(0, z) = f(z)$ and $H(1, z) = g(z)$ for all $z \in \mathscr{K}$.

THEOREM 2.9 (Borsuk). *Let $\mathscr{K} \subset \mathbb{C}$ be compact and let $f \colon \mathscr{K} \to \mathbb{C} \backslash \{0\}$ be continuous. Then the following statements are equivalent:*

(1) *there exists a homotopy in $\mathbb{C} \backslash \{0\}$ of f with a constant function.*

(2) *there exists a continuous function $g \colon \mathscr{K} \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathscr{K}$.*

(3) *f has an extension to a continuous function $F \colon \mathbb{C} \to \mathbb{C} \backslash \{0\}$.*

Proof. We will prove $(1) \Leftrightarrow (2)$ and $(2) \Leftrightarrow (3)$.

"$(1) \Rightarrow (2)$" Let H be a homotopy of f in $\mathbb{C} \backslash \{0\}$ with a constant α. By the uniform continuity of H on $\mathscr{K}$, $\lim_{t \to s} \|H(t, \cdot) - H(s, \cdot)\|_\infty = 0$. Hence H is a path in $\operatorname{inv}(\mathscr{C}(\mathscr{K}))$ from f to αe. Now (2) follows from Theorem 1.3.

"(2) $\Rightarrow$ (1)" If g is any function as in (2), then $H(t, z) := e^{(1-t)g(z)}$ is a homotopy in $\mathbb{C}\setminus\{0\}$ of f with the constant function 1.

"(2) $\Rightarrow$ (3)" Let g be any function as in (2). By the Tietze extension theorem [25, Th. 20.4], g has a continuous extension G to $\mathbb{C}$. It is obvious that the function e^G is a non-vanishing continuous extension of f to $\mathbb{C}$.

"(3) $\Rightarrow$ (2)" Since $\mathscr{K}$ is compact, there is $r > 0$ such that $\mathscr{K} \subset \{z \in \mathbb{C} \mid |z| \leqslant r\}$. By Example 2.4(a) and Theorem 2.5, there exists a continuous function g such that $F(z) = e^{g(z)}$ for all $|z| \leqslant r$. ∎

Remark 2.10. If $\mathscr{K}$ is a compact connected subset of $\mathbb{C}$ with connected complement, then every non-vanishing continuous function on $\mathscr{K}$ satisfies (1) of Theorem 2.9 (R. J. Fokkink showed the author that this is a special case of the Alexander duality theorem [28, Chap. 11]; there seems to be no direct simple proof of this special case). Hence every non-vanishing continuous function on $\mathscr{K}$ has a continuous logarithm (cf. Remark 2.6(b)).

(c) Let $\mathscr{A}_r$ $(r > 0)$ be the Banach algebra of all continuous functions on $\{z \in \mathbb{C} : |z| \leqslant r\}$ that are holomorphic on $\{z \in \mathbb{C} : |z| < r\}$. The norm is the supremum norm. If $r = 1$, then we write $\mathscr{A}$ for $\mathscr{A}_1$. The algebra $\mathscr{A}$ is known as the disc algebra.

THEOREM 2.11. *If $f \in \mathscr{A}_r$ does not vanish on $\{z \in \mathbb{C} : |z| \leqslant r\}$, then there is a $g \in \mathscr{A}_r$ such that $f(z) = e^{g(z)}$ for all $|z| \leqslant r$.*

Proof. We can use Theorem 2.5 or proceed as follows: the complex homomorphisms of $\mathscr{A}_r$ are point-evaluations on $\{z \in \mathbb{C} : |z| \leqslant r\}$ (see [25, proof of Th. 18.18]). Hence $f \in \operatorname{inv}(\mathscr{A}_r)$ and F, defined by $F(t)(z) := f(tz)$, is a path in $\operatorname{inv}(\mathscr{A}_r)$ from $f(0)e$ to f. We conclude from Theorem 1.3 that $f \in \exp(\mathscr{A}_r)$. ∎

Using the same trick as in the proof of Theorem 2.7, we now extend Theorem 2.11 to entire functions, i.e., functions holomorphic on $\mathbb{C}$.

THEOREM 2.12. *Let f be a non-vanishing entire function such that $f(0) = 1$. Then there exists a unique entire function g such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$ and $g(0) = 0$.*

Proof. Applying Theorem 2.11 to the Banach algebras $\mathscr{A}_n$ $(n \in \mathbb{N})$ and the restrictions f_n of f to $\{z \in \mathbb{C} : |z| \leqslant n\}$, we obtain holomorphic functions g_n on $\{z \in \mathbb{C} : |z| \leqslant n\}$ such that $g_n(0) = 0$ and $e^{g_n(z)} = f_n(z)$ for all $|z| \leqslant n$. It follows from Lemma 2.1 that the function g, defined by $g(z) := g_n(z)$ if $|z| \leqslant n$, is well defined. Clearly g is entire, $g(0) = 0$ and $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$. Unicity follows from Lemma 2.1. ∎

Remark 2.13. (a) For topological treatments of the theorems on functions of this section, see [8, Chap. IV] (uses homotopy) or [12, Chap. 1] (uses covering spaces).

(b) There is a deep theorem due to Arens and Royden (see, e.g., [13, pp. 88–91]) which says that the quotient group $\mathrm{inv}(\mathscr{B})/\exp(\mathscr{B})$ is isomorphic to $\mathscr{H}^1(\Delta(\mathscr{B}), \mathbb{Z})$, the first Čech cohomology group of $\Delta(\mathscr{B})$. From the Arens–Royden theorem we can, e.g., easily deduce Theorem 2.5: $\mathrm{inv}(\mathscr{B})/\exp(\mathscr{B}) \cong \mathscr{H}^1(\Delta(\mathscr{B}), \mathbb{Z}) \cong \mathscr{H}^1(\mathscr{K}, \mathbb{Z}) \cong \{0\}$, hence $\mathrm{inv}(\mathscr{B}) = \exp(\mathscr{B})$, since $\mathrm{inv}(\mathscr{B}) \subset \exp(\mathscr{B})$.

For more information on cohomology and Banach algebras, see the survey articles by Johnson and Taylor in [29].

3. Algebras on the Circle

(a) In this section we will derive analogues of Theorem 2.5 for $\mathscr{C}(\mathscr{T})$, where $\mathscr{T} := \{z \in \mathbb{C} : |z| = 1\}$, and the Wiener algebra. These results will be used to prove Theorem 3.7, which is essential for Section 4.

Let us have a closer look at $\mathrm{inv}(\mathscr{C}(\mathscr{T}))$ before stating and proving the correct analogue of Theorem 2.5. Note that $\mathscr{T}$ is not contractible [6] and that Theorem 2.5 is not true for $\mathscr{K} = \mathscr{T}$. For example, $e^{i\theta} \in \mathrm{inv}(\mathscr{C}(\mathscr{T}))$, but $e^{i\theta} \notin \exp(\mathscr{C}(\mathscr{T}))$ [6].

Let $f \in \mathrm{inv}(\mathscr{C}(\mathscr{T}))$ be arbitrary. Then f can be identified with a non-vanishing continuous function on $[-\pi, \pi]$, hence by Theorem 2.5 there exists a $g \in \mathscr{C}([-\pi, \pi])$ such that

$$f(e^{i\theta}) = e^{g(\theta)} \qquad \text{for all} \quad \theta \in [-\pi, \pi]. \tag{2}$$

Moreover, if $g_1, g_2 \in \mathscr{C}([-\pi, \pi])$ both satisfy (2), then an application of Lemma 2.1 to $g(\theta) := g_1(\theta) - g_1(-\pi) + g_2(-\pi)$ and $g(\theta) := g_2(\theta)$ yields $g_1(\pi) - g_1(-\pi) = g_2(\pi) - g_2(-\pi)$. Thus the following notion is well defined:

Definition 3.1. Let f be a non-vanishing continuous function on $\mathscr{T}$. Then $\mathrm{ind}(f)$, the *index of f*, is defined to be $(2\pi i)^{-1}\{g(\pi) - g(-\pi)\}$, where g is any continuous function on $[-\pi, \pi]$ satisfying (2).

Note that $\mathrm{ind}(f)$ is always an integer and $\mathrm{ind}(fg) = \mathrm{ind}(f) + \mathrm{ind}(g)$.

We can now state the analogues of Theorem 2.5 alluded to in the introduction of this section.

Theorem 3.2. $\exp(\mathscr{C}(\mathscr{T})) = \{f \in \mathrm{inv}(\mathscr{C}(\mathscr{T})) \mid \mathrm{ind}(f) = 0\}$.

Proof. The inclusion "$\subset$" is trivial. For the opposite inclusion, let f be an invertible element of $\mathscr{C}(\mathscr{T})$ with $\operatorname{ind}(f)=0$ and let g be a continuous function such that $f(e^{i\theta})=e^{g(\theta)}$ for all $\theta\in[-\pi,\pi]$. It follows from $\operatorname{ind}(f)=0$ that $g(\pi)=g(-\pi)$. Hence G, defined by $G(e^{i\theta}):=g(\theta)$, belongs to $\mathscr{C}(\mathscr{T})$ and $f=e^G$. ∎

It was mentioned in the introduction of this section that $e^{i\theta}\notin\exp(\mathscr{T})$. This follows directly from Theorem 3.2, since $\operatorname{ind}(e^{i\theta})=1$. Hence Theorem 2.5 implies that $\mathscr{T}$ is not contractible.

The following theorem describes the components of $\operatorname{inv}(\mathscr{C}(\mathscr{T}))$:

THEOREM 3.3. *Define* $C_k:=\{f\in\operatorname{inv}(\mathscr{C}(\mathscr{T})):\operatorname{ind}(f)=k\}$ $(k\in\mathbb{Z})$. *The components of* $\operatorname{inv}(\mathscr{C}(\mathscr{T}))$ *are precisely the sets* C_k.

Proof. It is clear that the sets C_k form a partition of $\operatorname{inv}(\mathscr{C}(\mathscr{T}))$. Note that $\mathscr{C}_0$ is a component by Theorem 1.2 and that $C_0=\exp(\mathscr{C}(\mathscr{T}))$ by Theorem 3.2. Hence the theorem holds for $k=0$. Define for each $k\in\mathbb{Z}$ the map F_k by $(F_kf)(e^{it}):=e^{itk}f(e^{it})$. It follows from $\operatorname{ind}(e^{itk})=k$ and $\operatorname{ind}(fg)=\operatorname{ind}(f)+\operatorname{ind}(g)$ that F_k maps C_0 onto C_k. Moreover, it is easy to see that F_k is an isometry. Hence the sets C_k $(k\neq 0)$ are homeomorphic with C_0, and we are done. ∎

(b) The *Wiener algebra* (denoted by $\mathscr{W}$) consists of all continuous functions on $\mathscr{T}:=\{z\in\mathbb{C}:|z|=1\}$ that have absolutely convergent Fourier series (see [24, Example 11.13(b)]).

Addition and multiplication are defined pointwise; the norm is defined by $\|\sum_{n=-\infty}^{\infty}a_ne^{in\theta}\|:=\sum_{n=-\infty}^{\infty}|a_n|$. Note that the algebra $l_1(\mathbb{Z})$ of absolutely summable two-sided sequences is isometric to $\mathscr{W}$. A famous theorem by Wiener states that the invertible elements of $\mathscr{W}$ are precisely those elements of $\mathscr{W}$ that do not vanish on $\mathscr{T}$ (see, e.g., [25, Lemma 11.6]).

The analogue of Theorem 2.5 for the Wiener algebra $\mathscr{W}$ can be found in [9]. We state this result as Theorem 3.4 and remark that the proof in [9] uses a special case of the deep Wiener–Lévy theorem [24, Th. 10.27].

THEOREM 3.4. $\exp(\mathscr{W})=\{f\in\operatorname{inv}(\mathscr{W})\mid\operatorname{ind}(f)=0\}$.

Proof. See [9, p. 491, Lemma] ∎

THEOREM 3.5. *Define* $\mathscr{M}_k:=\{f\in\operatorname{inv}(\mathscr{W}):\operatorname{inv}(f)=k\}$ $(k\in\mathbb{Z})$. *The components of* $\operatorname{inv}(\mathscr{W})$ *are precisely the sets* C_k.

Proof. The proof is identical to the proof of Theorem 3.3. ∎

We conclude this section with a result (Theorem 3.7) which will be essential for one of the main theorems of this paper (Theorem 4.3). Theorem 3.7 is an extension of Theorem 2.2.

LEMMA 3.6. *Let* $f, g \in \operatorname{inv}(\mathscr{C}(\mathscr{T}))$ *be such that* $|f(z)-g(z)|<|f(z)|$ *for all* $z \in \mathscr{T}$. *Then* $\operatorname{ind}(f)=\operatorname{ind}(g)$.

Proof. Note that $\operatorname{ind}(g)=\operatorname{ind}(f)+\operatorname{ind}(g/f)$. The assumptions imply that $|1-g(z)/f(z)|<1$ for all $z \in \mathscr{T}$. Hence $g/f \in \operatorname{inv}(\mathscr{C}(\mathscr{T}))$ and Theorem 2.5 yields a $\varphi \in \mathscr{C}([-\pi, \pi])$ such that $g(e^{i\theta})/f(e^{i\theta})=e^{i\varphi(\theta)}$ for all $\theta \in [-\pi, \pi]$. We deduce from $|1-g(e^{i\theta})/f(e^{i\theta})|<1$ that for all $\theta \in [-\pi, \pi]$, $\varphi(\theta) \in [-\pi/2, \pi/2]+2\pi k(\theta)$ with $k(\theta) \in \mathbb{Z}$. It follows from an elementary connectivity argument that $k(\theta)$ does not depend on θ, hence $\varphi(\pi)-\varphi(-\pi)=0$, i.e., $\operatorname{ind}(g/f)=0$. ∎

For notation of the following theorem, see Section 2(a).

THEOREM 3.7. $\{x \in l_1(\alpha)$: $\sum_{n=0}^{\infty} x_n e^{n\rho} e^{in\theta} \neq 0$ *for all* $\theta \in [-\pi, \pi]$ *and has index* 0 *on* $[-\pi, \pi]\} = \operatorname{inv}(l_1(\alpha)) = \exp(l_1(\alpha))$.

Proof. The last equality follows from Theorem 2.2. Let $x \in l_1(\alpha)$ be such that $\sum_{n=0}^{\infty} x_n e^{n\rho} e^{in\theta} \neq 0$ for all $\theta \in [-\pi, \pi]$ and has index 0 on $[-\pi, \pi]$. Define functions f and f_r $(0<r<1)$ on $\mathscr{D}$ by $f(z):=\sum_{n=0}^{\infty} x_n (ze^{\rho})^n$ and $f_r(z):=f(rz)$. Since f is a non-vanishing continuous function on $\mathscr{T}$, there exists $\delta>0$ such that $\delta<|f(z)|$ for all $z \in \mathscr{T}$. If $r<1$ is close enough to 1, then $|f(z)-f_r(z)|<\delta<|f(z)|$ for all $z \in \mathscr{T}$. By Lemma 3.6, $\operatorname{ind}(f_r)= \operatorname{ind}(f)=0$ (restrict f_r and f to $\mathscr{T}$). Now the Argument Principle [8, p. 179, Cor. 5.86] yields that $\sum_{n=0}^{\infty} x_n z^n \neq 0$ for $|z| \leqslant re^{\rho}$. Since r can be arbitrarily close to 1, $\sum_{n=0}^{\infty} x_n z^n \neq 0$ for $|z|<e^{\rho}$. Hence $x \in \operatorname{inv}(l_1(\alpha))$ by Theorem 2.2.

Conversely, if $x \in \operatorname{inv}(l_1(\alpha))$, then $\sum_{n=0}^{\infty} x_n z^n \neq 0$ for all $|z| \leqslant e^{\rho}$. A similar use of the Argument Principle as above yields that $\sum_{n=0}^{\infty} x_n e^{n\rho} e^{in\theta}$ has index 0 on $[-\pi, \pi]$. ∎

4. APPLICATIONS TO POLYNOMIALS OF CONVOLUTION TYPE

In this section analytical results concerning the following generating function for polynomials of convolution type (see introduction) will be given ($g(z)$ denotes the formal power series $\sum_{k=0}^{\infty} g_k z^k$):

$$\sum_{n=0}^{\infty} q_n(t)\, z^n = e^{t \cdot g(z)}. \tag{3}$$

Formula (3) follows directly from Lemma 4.1 below. For the original proof of (3), see [20, p. 189, Cor. 2].

Notation. If $(g_n)_{n \in \mathbb{N}}$ is a sequence of complex numbers, then $g_n^{k*} := (g^{k*})_{n \in \mathbb{N}}$ is the k-fold convolution of $(g_n)_{n \in \mathbb{N}}$; g_n^{0*} is defined to be 1 for $n=0$ and 0 for $n>0$.

LEMMA 4.1. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. Then there is a unique sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $g_0=0$ such that*

$$q_n(t)=\sum_{k=0}^{n} g_n^{k*}\frac{t^k}{k!}. \tag{4}$$

Proof. See [7, Theorem 2.6]. ∎

Note that (4) implies that (1) holds for all $x, y\in\mathbb{C}$.

LEMMA 4.2. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. If $(\tilde{q}_n)_{n\in\mathbb{N}}$ is defined by $\tilde{q}_n(z):=\lambda^n q_n(z)$, then $(\tilde{q}_n)_{n\in\mathbb{N}}$ is also a sequence of polynomials of convolution type with corresponding $\tilde{g}_n=\lambda^n g_n$.*

Proof. This follows directly from (1) and (3). ∎

Notation. If $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type, we will write $\psi(t,z):=\sum_{n=0}^{\infty} q_n(t)\,z^n$ whenever this series converges absolutely.

THEOREM 4.3. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. Then the following are equivalent*:

(a) $(g_n)_{n\in\mathbb{N}}\in l_1$.

(b) *There are M, $\delta>0$ such that* $\|(q_n(t))_{n\in\mathbb{N}}\|_1\leqslant e^{tM}$ for $0<t<\delta$.

(c) $\lim_{t\downarrow 0}\|q_n(t))_{n\in\mathbb{N}}\|_1=1$.

(d) $\limsup_{t\downarrow 0}\|(q_n(t))_{n\in\mathbb{N}}\|_1<2$.

(e) *There are $\delta>0$ and $0<t_0<\delta$ such that $(q_n(t))_{n\in\mathbb{N}}\in l_1$ for all $t\in(0,\delta)$ and $\psi(t_0,z)\neq 0$ if $|z|=1$.*

(f) *There is $t\in\mathbb{C}\setminus\{0\}$ such that $(q_n(t))_{n\in\mathbb{N}}\in l_1$ and $\psi(t,z)\neq 0$ if $|z|\leqslant 1$.*

(g) *There is $t\in\mathbb{C}\setminus\{0\}$ such that $(q_n(t))_{n\in\mathbb{N}}\in l_1$ and $(q_n(-t))_{n\in\mathbb{N}}\in l_1$.*

Moreover, if one of these conditions holds, then (3) holds and both series in (3) converge absolutely for all $t\in\mathbb{C}$ and $|z|\leqslant 1$.

Proof. "(a) ⇒ (b)" $e^{t\,\|(g_n)_{n\in\mathbb{N}}\|_1}=\sum_{k=0}^{\infty}(\|(g_n)_{n\in\mathbb{N}}\|_1^k\,(t^k/k!))\geqslant 1+\sum_{k=1}^{\infty}(\sum_{n=k}^{\infty}|g_n^{k*}|\;(t^k/k!))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}|g_n^{k*}|\;(t^k/k!)\geqslant 1+\sum_{n=1}^{\infty}|\sum_{k=1}^{n} g_n^{k*}(t^k/k!)|=\sum_{n=0}^{\infty}|q_n(t)|$.

"(b) ⇒ (c)" This follows from $q_0\equiv 1$.

"(c) ⇒ (d)" This is trivial.

"(d) ⇒ (e)" There exists t_0 such that $\|(q_n(t_0))_{n\in\mathbb{N}}\|_1<2$, hence $|\psi(t_0,z)|-1<2-1=1$ for all $|z|=1$.

"(e) ⇒ (f)" Note that the assumptions imply that $\psi(t, z) \neq 0$ for all $t \in (0, \delta)$ and $|z| = 1$ by [16, p. 144, Th. 4.17.1]. Recall that the index of a non-vanishing continuous function on $\mathscr{T}$ (Definition 3.1) is always an integer. It is clear that for fixed t, s: $\operatorname{ind}(\psi(t, \cdot)) + \operatorname{ind}(\psi(s, \cdot)) = \operatorname{ind}(\psi(t+s, \cdot))$. If $\operatorname{ind}(\psi(t, \cdot)) \neq 0$ for some $t \in (0, \delta)$, then $\operatorname{ind}(\psi(t/n, \cdot)) \notin \mathbb{Z}$ for n large enough, which is impossible. Hence Theorems 2.2 and 3.7 imply that for all $t \in (0, \delta)$, $\psi(t, z) \neq 0$ if $|z| \leqslant 1$.

"(f) ⇒ (g)" By Theorem 2.2, there exists a unique sequence $(a_k)_{k \in \mathbb{N}} \in l_1$ such that $\sum_{k=0}^{n} q_k(x)\, a_{n-k} = \delta_{0n}$. Comparing coefficients and using (1) we see that each a_k must be equal to $q_k(-t)$.

"(g) ⇒ (a)" It follows from (1) that $\psi(t, z)\, \psi(-t, z) = \psi(0, z) = 1$ for all $|z| \leqslant 1$. Hence $\psi(t, z) \neq 0$ for all $|z| \leqslant 1$. By Theorem 2.2 there exists a sequence $b = (b_n)_{n \in \mathbb{N}} \in l_1$ such that $(q_n(t))_{n \in \mathbb{N}} = e^b$. Comparing coefficients and using Lemma 4.1, we obtain $b_n = t g_n$ for all $n \in \mathbb{N}$ and $(g_n)_{n \in \mathbb{N}} \in l_1$.

The last statement follows from the uniqueness of Taylor coefficients of analytic functions and the fact that (3) implies that (b) actually holds for all $t \in \mathbb{C}$. ∎

Remark 4.4. (1) It follows from the proof of "(a) ⇒ (b)" that $M = \|(g_n)_{n \in \mathbb{N}}\|_1$ suffices in (b).

(2) If (b), (f), or (g) holds, then (3) implies that they hold for all $t \in \mathbb{C}$.

(3) If $g_n = 0$ for n even an $(g_n)_{n \in \mathbb{N}} \notin l_1$, then $(q_n(t))_{n \in \mathbb{N}} \notin l_1$ for any $t \neq 0$ since $q_n(-t) = (-1)^n q_n(t)$.

(4) For an alternative proof of "(e) ⇒ (f)" see Remark 5.3.

(5) We may weaken (d) to (d'): "there is $t \in \mathbb{C} \backslash \{0\}$ such that $\|(q_n(t))_{n \in \mathbb{N}}\|_1 < 2$," since obviously (c) ⇒ (d') ⇒ (f).

We will now look at the relation between the radii of convergence of $\sum_{n=0}^{\infty} q_n(t)\, z^n$ and $\sum_{k=0}^{\infty} g_k z^k$.

Notation.

$$\mathscr{R}_g := \text{radius of convergence of } \sum_{k=0}^{\infty} g_k z^k.$$

$$\rho_t := \text{radius of convergence of } \sum_{n=0}^{\infty} q_n(t)\, z^n.$$

$$\mathscr{N}_t := \{z : |z| < \rho_t \text{ and } \psi(t, z) = 0\}.$$

$$v_t := \inf\{|z| : z \in \mathscr{N}_t\} \text{ if } \mathscr{N}_t \neq \varnothing,\ v_t := \rho_t \text{ if } \mathscr{N}_t = \varnothing.$$

We start our discussion with some examples.

EXAMPLES 4.5. (a) $q_n(t)=x^n/n! : \rho_t=\mathscr{R}_g=\infty$ for all $t\in\mathbb{C}$.

(b) $q_n(t)=\binom{t}{n} : \mathscr{R}_g=1$; $\rho_t=\infty$ for $t\in\mathbb{N}$, $\rho_t=1$ for $t\notin\mathbb{N}$.

(c) $q_n(t)=t(t-an)^{n-1}/n! : \rho_t=\mathscr{R}_g=(|a|\,e)^{-1}$ (use Stirling's formula).

Note that $\rho_0=\infty$ because $q_n(0)=0$ for $n\geqslant 1$.

It follows from Theorem 4.3(a) $\Rightarrow$ (b) and Lemma 4.2 that $\mathscr{R}_g\leqslant\rho_t$ for all $t\in\mathbb{C}$. The examples suggest that $\mathscr{R}_g=\rho_t$ for all except countably many t. Remark 4.4(5) shows that it is not possible that $\rho_t>\mathscr{R}_g$ for all $t>0$. The following theorem shows that the zeros of the functions $\psi(t,\cdot)$ determine $\mathscr{R}_g$. Moreover, it enables us to prove the important property stated as part (e) of Theorem 4.6. This property will play an important role in the rest of this section and in Section 5.

THEOREM 4.6. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. Then:*

(a) *If $\mathscr{R}_g=0$, then $\rho_t=0$ for all $t\in\mathbb{C}\setminus\{0\}$.*

(b) *If $\mathscr{R}_g=\infty$, then $\rho_t=\infty$ for all $t\in\mathbb{C}\setminus\{0\}$.*

(c) *If $0<\mathscr{R}_g<\infty$, then $\mathscr{R}_g=v_t$ for all $t\in\mathbb{C}\setminus\{0\}$. In particular, $v_t=v_s$ for all t, $s\in\mathbb{C}\setminus\{0\}$ and $\psi(t,z)\neq 0$ for all $t\in\mathbb{C}$ and all $|z|<\mathscr{R}_g$.*

(d) *There are at most countably many $t\in\mathbb{C}$ such that $\rho_t>\mathscr{R}_g$.*

(e) *If $|z|<\rho_t$ for uncountably many $t\in\mathbb{C}$, then $\psi(t,z)\neq 0$ for all $t\in\mathbb{C}$.*

Proof. (a) Suppose $\rho_t\neq 0$ for some $t\neq 0$. Since $\psi(t,0)=1$ there is a δ, $0<\delta<\rho_t$, such that $|\psi(t,z)|>0$ for $|z|\leqslant\delta$. Now Lemma 4.2 and Theorem 4.3(f) $\Rightarrow$ (a) imply that $(g_n\delta^n)_{n\in\mathbb{N}}\in l_1$, hence $\mathscr{R}_g\geqslant\delta>0$.

(b) This follows from Lemma 4.2 and Theorem 4.3(a) $\Rightarrow$ (b).

(c) Let $t\in\mathbb{C}\setminus\{0\}$ be arbitrary. We first prove $\mathscr{R}_g\leqslant v_t$. If $|z|<\mathscr{R}_g$, then $(g_nz^n)_{n\in\mathbb{N}}\in l_1$. It follows from Lemma 4.2 and Theorem 4.3 that $\psi(t,z)=e^{t\cdot g(z)}\neq 0$ for $|z|<\mathscr{R}_g$. Hence $\mathscr{R}_g\leqslant v_t$.

The reverse inequality $\mathscr{R}_g\geqslant v_t$ follows from Theorem 4.3(f) $\Rightarrow$ (a) and Remark 4.4(2).

(d) Suppose $\rho_s>\mathscr{R}_g$ for some $s\in\mathbb{C}\setminus\{0\}$. Note that $\psi(s,\cdot)$ has at least one zero on $|z|=\mathscr{R}_g$ by Theorem 4.3(f) $\Rightarrow$ (a). Therefore we may write $\psi(s,z)=f_s(z)\ \prod_{j=1}^k(1-\alpha_jz)^{r_j}$ with $|1/\alpha_j|=\mathscr{R}_g$ and $r_j\in\mathbb{N}$, $j=1,\dots,k$. There exists $\delta>0$ such that $\psi(s,\cdot)$ has finitely many zeros on $\{z:\mathscr{R}_g\leqslant|z|\leqslant\mathscr{R}_g+\delta<\rho_s\}$. Hence f_s is a non-vanishing analytic function on $|z|\leqslant\mathscr{R}_g+\delta_1$ for some $\delta_1>0$. Since $f_s(0)=1$, Theorem 2.11 yields a unique analytic function h such that $h(0)=0$ and $f_s(z)=e^{h(z)}$ for $|z|<\mathscr{R}_g+\delta_1$. Hence $\psi(s,z)=\exp\{h(z)+\sum_{j=1}^k r_j\operatorname{Log}(1-\alpha_jz)\}$ for $|z|<\mathscr{R}_g$. By

Lemma 2.1 and Theorem 4.3, $h(z)+\sum_{j=1}^{k} r_j \operatorname{Log}(1-\alpha_j z)=sg(z)$ for $|z|<\mathcal{R}_g$. It follows that $\psi(t,z)=e^{(t/s)s\cdot g(z)}=\exp\{t/s\, h(z)+t/s \sum_{j=1}^{k} r_j \operatorname{Log}(1-\alpha_j z)\}=\exp(t/s\, h(z)) \prod_{j=1}^{i} (1-\alpha_j z)^{r_j t/s}$ for all $t\in\mathbb{C}$ and all $|z|<\mathcal{R}_g$. We conclude from the analyticity of h on $|z|<\mathcal{R}_g+\delta_1$, that $\rho_t>\mathcal{R}_g$ if and only if $t/s\in\mathbb{N}$.

(e) It follows from (d) that $|z|<\mathcal{R}_g$, hence $\psi(t,z)\neq 0$ by (c). ∎

The Banach algebra $\mathcal{TA}$ consists of all one-sided sequences $(a_n)_{n\in\mathbb{N}}$ of complex numbers such that $f(z):=\sum_{n=0}^{\infty} a_n z^n$ is analytic on $\mathcal{D}$ and can be extended to a continuous function on $\bar{\mathcal{D}}$. Addition is defined componentwise, multiplication is defined to be convolution. The norm on $\mathcal{TA}$ is defined by $\|(a_n)_{n\in\mathbb{N}}\|_{\mathcal{TA}}:=\sup_{|z|<1} |\sum_{n=0}^{\infty} a_n z^n|$.

The space $\mathcal{TA}$ is isometric to the disc algebra $\mathcal{A}$ (see Section 2(c)).

THEOREM 4.7. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. Then the following two conditions are equivalent*:

(a) $(g_n)_{n\in\mathbb{N}}\in\mathcal{TA}$.

(b) *There are $t\in\mathbb{C}\setminus\{0\}$ and $\delta>0$ such that $(q_n(t))_{n\in\mathbb{N}}\in\mathcal{TA}$ and $|\psi(t,z)|>\delta$ for all $z\in\mathcal{D}$.*

If (a) *or* (b) *holds, then*:

(c) $(q_n(t))_{n\in\mathbb{N}}\in\mathcal{TA}$ *for all $t\in\mathbb{C}$ and $|\psi(t,z)|>\delta(t)>0$ for all $z\in\mathcal{D}$.*

(d) (3) *holds for all $t\in\mathbb{C}$ and all $z\in\mathcal{D}$.*

Proof. "(a) ⇒ (b)" By Lemma 4.2 and Theorem 4.3(a) ⇒ (e), (3) holds for all t, z with $t\geqslant 0$ and $|z|<1$. Hence $(q_n(t))_{n\in\mathbb{N}}\in\mathcal{TA}$ fort all $t\in\mathbb{C}$ since clearly $\exp\{(g_n)_{n\in\mathbb{N}}\}\in\mathcal{TA}$. The second statement of (b) also follows from (3).

"(b) ⇒ (a)" By Lemma 4.2 and Theorem 4.3(f) ⇒ (a), (3) holds for all $z\in\mathcal{D}$. Hence $\exp(t\sum_{k=0}^{\infty} g_k z^k)$ has a non-vanishing continuous extension G to $\bar{\mathcal{D}}$. By Theorem 2.11, there exists a unique function $h\in\mathcal{A}$ such that $h(0)=g_0=0$ and $G=e^{th}$. From $h(0)=0$ and $e^{th}=\exp\{t\sum_{k=0}^{\infty} g_k z^k\}$ we deduce that $\sum_{k=0}^{\infty} g_k z^k=h(z)$ for $z\in\mathcal{D}$, hence $g\in\mathcal{A}$ and $(g_n)_{n\in\mathbb{N}}\in\mathcal{TA}$.

Statements (c) and (d) follow easily from the proof of (a) ⇔ (b) and the uniqueness of Taylor coefficients of analytic functions. ∎

5. Applications to Two-Sided Sequences of Functions of Convolution Type

(a) In this section we will study a two-sided analogue of sequences of polynomials of convolution type.

DEFINITION. Let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of Lebesgue measurable functions on $[0, \infty)$ such that not all functions q_n are identically zero. Then $(q_n)_{n\in\mathbb{Z}}$ is said to be a *two-sided sequence of convolution type* if

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} |q_k(t)\, q_{n-k}(s)| < \infty && \text{for all} \quad s, t \geqslant 0 \\ &q_n(t+s) = \sum_{k=-\infty}^{\infty} q_k(t)\, q_{n-k}(s) && \text{for all} \quad s, t \geqslant 0. \end{aligned} \tag{5}$$

We write $\varphi(t, z) := \sum_{n=-\infty}^{\infty} q_n(t)\, z^n$ whenever this series converges absolutely. Note that $\varphi(t+s, z) = \varphi(t, z)\, \varphi(s, z)$.

Contrary to the one-side case, there seems to be no algebraic theory for two-sided sequences of convolution type. Therefore our policy is to impose several analytical conditions on $\varphi(t, z)$ and study the consequences.

For later use we state the following lemma.

LEMMA 5.1. *If $(c_n)_{n\in\mathbb{Z}}$ is a two-sided sequence of complex numbers and $(c_n R^n)_{n\in\mathbb{Z}} \in l_1(\mathbb{Z})$ for some $R > 0$, then $\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (t^k/k!)\, c_n^{k*} R^n$ converges absolutely for all $t \in \mathbb{C}$. In particular, $\sum_{k=0}^{\infty} (t^k/k!)\, c_n^{k*}$ converges absolutely for all $t \in \mathbb{C}$.*

Proof. This follows from $\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} |(t^k/k!)\, c_n^{k*} R^n| \leqslant \sum_{k=0}^{\infty} |(t^k/k!)| \sum_{n=-\infty}^{\infty} |c_n^{k*}|\, R^n| \leqslant \sum_{k=0}^{\infty} |(t^k/k!)|\, \{\sum_{n=-\infty}^{\infty} |c_n|\, R^n\}^k < \infty$. ∎

We begin with demanding $\varphi(t, \cdot)$ to be an invertible element of the Wiener algebra $\mathscr{W}$, i.e., $\varphi(t, z) \neq 0$ for all $z \in \mathscr{T}$ (cf. Section 3(c)) and [24, Example 11.13(b)]).

THEOREM 5.2. *Let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of convolution type. If $\varphi(t, \cdot) \in \operatorname{inv}(\mathscr{W})$ for all $t \geqslant 0$, then there exists an $h \in \mathscr{W}$ such that $\varphi(t, z) = e^{t \cdot h(z)}$ for all $|z| = 1$. In particular, there exists a two-sided sequence $(h_n)_{n\in\mathbb{Z}} \in l_1(\mathbb{Z})$ such that $q_n(t) = \sum_{j=0}^{\infty} h_n^{j*}(t^j/j!)$.*

Proof. Define φ_1 by $\varphi_1(t, \theta) := \varphi(t, e^{i\theta})$. From the measurability of $\varphi(\cdot, z)$ and [16, p. 145, Cor. 4.17.3] we obtain

$$\varphi_1(t, \theta)/\varphi_1(t, 0) = \exp(t\chi(\theta)). \tag{6}$$

Theorem 2.5 yields the existence of continuous functions $\gamma(t, \cdot)$ on $[-\pi, \pi]$ such that

$$\varphi_1(t, \theta)/\varphi_1(t, 0) = \exp(\gamma(t, \theta)) \tag{7}$$

with $\gamma(t, 0) = 0$. We obtain from (6) and (7)

$$\gamma(t, \theta) = t\chi(\theta) + k(t, \theta)\, 2\pi i \tag{8}$$

with $k(t, \theta) \in \mathbb{Z}$. From (8) with $t = 1$ we obtain

$$\gamma(t, \theta) = t\gamma(1, \theta) + \{k(t, \theta) - tk(1, \theta)\}\, 2\pi i. \tag{9}$$

From (9), the continuity of $\gamma(t, \cdot)$, and $\gamma(t, 0) = 0$ we obtain $k(t, \theta) - tk(1, \theta) = 0$, hence

$$\varphi_1(t, \theta)/\varphi_1(t, 0) = \exp(t\gamma(1, \theta)). \tag{10}$$

From (10) and the measurability of $\varphi_1(t, 0)$ we have

$$\varphi_1(t, \theta) = \exp\{at + t\gamma(1, \theta)\}. \tag{11}$$

From $\varphi_1(t, \pi) = \varphi_1(t, -\pi)$ we obtain $t\gamma(1, \pi) = t\gamma(1, -\pi) + k(t)\, 2\pi i$ with $k(t) \in \mathbb{Z}$, for all $t \geqslant 0$. Hence $k(t) = 0$ and $\gamma(1, \pi) = \gamma(1, -\pi)$. An application of Theorem 3.4 to (11) yields

$$at + t\gamma(1, \theta) = \sum_{n=-\infty}^{\infty} h_n(t)\, e^{in\theta} \qquad \text{with} \qquad \sum_{n=-\infty}^{\infty} |h_n(t)| < \infty. \tag{12}$$

Since $h_n(t) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-in\theta}\, (at + t\gamma(1, \theta))\, d\theta$, we must have $h_n(t) = th_n$. We conclude that $\varphi(t, e^{i\theta}) = \exp\{t \sum_{n=-\infty}^{\infty} h_n e^{in\theta}\}$ with $\sum_{n=-\infty}^{\infty} |h_n| < \infty$. ∎

Remark 5.3. Using Theorem 5.2 we can give an interesting proof of Theorem 4.3(e) $\Rightarrow$ (f) as follows: Extend the sequences $(q_n(t))_{n \in \mathbb{N}}$ to elements of $l_1(\mathbb{Z})$ by setting $q_n(t) = 0$ for $n < 0$. By Theorem 5.2, there exists a sequence $(c_n)_{n \in \mathbb{Z}} \in l_1(\mathbb{Z})$ such that

$$\sum_{n=-\infty}^{\infty} q_n(t)\, e^{in\theta} = \exp\left\{t \sum_{n=-\infty}^{\infty} c_n e^{in\theta}\right\} = \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{k=0}^{\infty} \frac{t^k}{k!}\, c_n^{k*}$$

(Lemma 5.1 allows us to change the order of summation). Unicity of Fourier coefficients yields

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{t^k}{k!}\, c_n^{k*} = 0 \qquad && \text{for} \quad n < 0; \\ &\sum_{k=0}^{\infty} \frac{t^k}{k!}\, c_n^{k*} = q_n(t) \qquad && \text{for} \quad n \geqslant 0 \end{aligned} \tag{13}$$

Because (13) holds for all $t \geqslant 0$, we must have $c_n = 0$ for $n < 0$. Hence

$$\sum_{n=0}^{\infty} q_n(t) z^n = \exp\left\{t \sum_{n=0}^{\infty} c_n z^n\right\} \neq 0 \qquad \text{for all} \quad |z| \leqslant 1. \tag{14}$$

Notation. If $0 < a < b < \infty$, then $\mathscr{A}(a, b) := \{z \in \mathbb{C} : a < |z| < b\}$ and $\bar{\mathscr{A}}(a, b) := \{z \in \mathbb{C} : a \leqslant |z| \leqslant b\}$.

(b) We denote the Banach algebra of all Laurent series that converge absolutely on $\bar{\mathscr{A}}(a, b)$ by $\mathscr{W}_{a,b}$. Addition and multiplication are the usual addition and multiplication of series. The norm on $\mathscr{W}_{a,b}$ is defined by

$$\left\| \sum_{n=-\infty}^{\infty} x_n z^n \right\| := \max\left\{ \sum_{n=-\infty}^{\infty} |x_n| \, a^n, \sum_{n=-\infty}^{\infty} |x_n| \, b^n \right\}.$$

It is easy to see that $\mathscr{W}_{a,b}$ is complete and that the polynomials in z and $1/z$ are dense in $\mathscr{W}_{a,b}$.

The unit of $\mathscr{W}_{a,b}$ is the Laurent series with $x_0 = 1$ and $x_n = 0$ for $n \neq 0$.

LEMMA 5.4. *The complex homomorphisms of $\mathscr{W}_{a,b}$ are point-evaluations on $\bar{\mathscr{A}}(a, b)$.*

Proof. Let $\Lambda \in \Delta(\mathscr{W}_{a,b})$ be arbitrary. From $\|\Lambda\| = 1$ we infer for the polynomial z that $|\Lambda(z)| \leqslant b$ and $|\Lambda(1/z)| \leqslant \|\Lambda\| \; \|z^{-1}\| \leqslant \|z^{-1}\| = a^{-1}$. Since $1/z$ is inverse to z, $|\Lambda(z)| = 1/|\Lambda(1/z)| \geqslant a$. Hence if p is a polynomial in z and $1/z$, then $\Lambda(p) = p(\Lambda(z)) = p(z_0)$ for some $z_0 \in \bar{\mathscr{A}}(a, b)$. Since the polynomials in z and $1/z$ are dense in $\mathscr{W}_{a,b}$, we conclude that $\Lambda(f) = f(z_0)$ for every $f \in \mathscr{W}_{a,b}$. ∎

THEOREM 5.5. *Let $a, b \in \mathbb{R}$ $(a < b)$ and let $(q_n)_{n \in \mathbb{Z}}$ be a two-sided sequence of convolution type. If $\varphi(t, \cdot) \in \operatorname{inv}(\mathscr{W}_{a,b})$ for all $t \geqslant 0$, then there exists a Laurent series $\sum_{n=-\infty}^{\infty} g_n z^n \in \mathscr{W}_{a,b}$ such that $\varphi(t, z) = \exp\{t \sum_{n=-\infty}^{\infty} g_n z^n\}$ for all $z \in \bar{\mathscr{A}}(a, b)$. In particular, $q_n(t) = \sum_{k=0}^{\infty} g_n^{k*}(t^k/k!)$.*

Proof. Let $r \in [a, b]$ be arbitrary. Then $\varphi(t, re^{i\theta}) \neq 0$ for $\theta \in [-\pi, \pi]$. By Theorem 5.2 there exists $(c_n(r))_{n \in \mathbb{Z}} \in l_1(\mathbb{Z})$ such that

$$\varphi(t, re^{i\theta}) = \exp\left\{t \sum_{n=-\infty}^{\infty} c_n(r) \, e^{in\theta}\right\}. \tag{15}$$

Define $g_n(r) := c_n(r) \, r^{-n}$. Then $(g_n r^n)_{n \in \mathbb{Z}} \in l_1(\mathbb{Z})$. We will now prove that $g_n(r)$ does not depend on r. By Lemma 5.1, we may change the order of summation in (15) which yields $q_n(t) = \sum_{k=0}^{\infty} g_n(r)^{k*} \, (t^k/k!)$. Since r was arbitrary and the right-hand side series defines a holomorphic function of t, we conclude that $g_n(r)$ does not depend on r. Define $g_n := g_n(a)$. Hence

q_n has the form indicated above. Moreover, $(g_n r^n)_{n\in\mathbb{Z}} \in l_1(\mathbb{Z})$ for all $r \in [a, b]$ and thus (15) yields $\varphi(t, z) = \exp\{t \sum_{n=-\infty}^{\infty} g_n z^n\}$ for all $z \in \mathscr{A}(a, b)$. ∎

We now set out to prove the analogue of Theorem 5.5 for the open annulus. It turns out that two-sided sequences of convolution type possess a property that is analogous to the property for polynomials of convolution type as expressed in Theorem 4.6(e). This property is stated in Theorem 5.7; the above mentioned analogue of Theorem 5.5 is Theorem 5.8.

LEMMA 5.6. *Let $(g_n)_{n\in\mathbb{Z}}$ be an arbitrary double-sided sequence of complex numbers such that $\sum_{n=-\infty}^{\infty} g_n z^n$ converges absolutely on the open annulus $\mathscr{A}(a, b)$. Define $q_n(t) := \sum_{k=0}^{\infty} g_n^{k*}(t^k/k!)$ for $t \in [0, \infty)$. Then $(q_n)_{n\in\mathbb{Z}}$ is a two-sided sequence of convolution type. If $z \in \mathscr{A}(a, b)$, then $\sum_{n=-\infty}^{\infty} q_n(t)\, z^n$ converges absolutely and does not vanish.*

Proof. It follows from Lemma 5.1 that q_n $(n \in \mathbb{Z})$ is well defined and that $\sum_{n=-\infty}^{\infty} q_n(t)\, z^n$ converges absolutely on $\mathscr{A}(a, b)$. Hence $\sum_{n=-\infty}^{\infty} q_n(t)\, z^n = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} g_n^{k*}(t^k/k!)\, z^n = \sum_{k=0}^{\infty} \{\sum_{n=-\infty}^{\infty} g_n^{k*} z^n\}\, (t^k/k!) = \exp\{t \sum_{n=-\infty}^{\infty} g_n z^n\} \neq 0$. ∎

THEOREM 5.7. *Let $a, b \in \mathbb{R}$ $(a < b)$ and let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of convolution type. If $\varphi(t, z)$ converges for all $z \in \mathscr{A}(a, b)$ and all $t \geqslant 0$, then $\varphi(t, z) \neq 0$ for all $t \geqslant 0$ and all $z \in \mathscr{A}(a, b)$.*

Proof. Suppose there are $t_0 > 0$ and $z_0 \in \mathscr{A}(a, b)$ such that $\varphi(t_0, z_0) = 0$. It follows from [16, p. 144, Th. 4.17.1] that $\varphi(t, z_0) = 0$ for all $t > 0$. Choose c, d with $a \leqslant c < |z_0| < d \leqslant b$ such that $\varphi(t_0, z) \neq 0$ for all $t > 0$ and all $z \in \mathscr{A}(c, |z_0|) \cup \mathscr{A}(|z_0|, d)$. This is possibly since the functions $\varphi(t, \cdot)$ are analytic and not identically zero. Choose c_1, c_2, d_1, and d_2 with $c < c_1 < c_2 < |z_0|$ and $|z_0| < d_1 < d_2 < d$. An application of Theorem 5.5 to the functions q_n on $\bar{\mathscr{A}}(c_1, c_2)$ yields a sequence $(g_n)_{n\in\mathbb{N}}$ of complex numbers such that $q_n(t) = \sum_{k=0}^{\infty} g_n^{k*}(t^k/k!)$. An application of Theorem 5.5 to the functions q_n on $\bar{\mathscr{A}}(d_1. d_2)$ yields a sequence $(h_n)_{n\in\mathbb{N}}$ of complex numbers such that $q_n(t) = \sum_{k=0}^{\infty} h_n^{k*}(t^k/k!)$. Differentiating to t and substituting $t = 0$, we obtain $g_n = h_n$ for all $n \in \mathbb{Z}$. This implies that $\sum_{n=-\infty}^{\infty} g_n z^n$ converges for all $z \in \bar{\mathscr{A}}(c_1, c_2) \cup \bar{\mathscr{A}}(d_1, d_2)$, hence for all $z \in \bar{\mathscr{A}}(c_1, d_2)$. It follows from Lemma 5.6 that $\varphi(t_0, z) \neq 0$ for all $z \in \bar{\mathscr{A}}(c_1, d_2)$, which contradicts $\varphi(t_0, z_0) = 0$. ∎

THEOREM 5.8. *Let $a, b \in \mathbb{R}$ $(a < b)$ and let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of convolution type. Suppose $\varphi(t, z)$ converges absolutely for all $t \geqslant 0$ and all $z \in \mathscr{A}(a, b)$. Then there exists a Laurent series $\sum_{n=-\infty}^{\infty} g_n z^n$ that* *absolutely converges on $\mathscr{A}(a, b)$ and satisfies $\varphi(t, z)=\exp\{t \sum_{n=-\infty}^{\infty} g_n z^n\}$ for all $t \geqslant 0$ and for all $z \in \mathscr{A}(a, b)$. In particular, $\varphi(t, z)$ does not vanish on $\mathscr{A}(a, b)$ and $q_n(t)=\sum_{k=0}^{\infty} g_n^{k*}(t^k/k!)$.*

Proof. It follows from Theorem 5.7 that $\varphi(t, z) \neq 0$ for all $t \geqslant 0$ and all $z \in \mathscr{A}(a, b)$. Applying Theorem 5.5 to $\mathscr{W}_{a+1/n, b-1/n}$ for all $n \in \mathbb{N}$ we obtain Laurent series $h_n \in \mathscr{W}_{a+1/n, b-1/n}$ such that $\varphi(t, z)=\exp(th_n(z))$. Since $\exp(th_n(z))=\exp(th_m(z))$ for all $t \in [0, \infty)$ on a circular region, $h_n(z)=h_m(z)$ for all z in their common domain. Hence all the Laurent series h_n are identical. If we set $g := h_1$, then g converges absolutely on $\mathscr{A}(a, b)$ and $\varphi(t, z)=e^{t \cdot g(z)}$ for all $t \in \mathbb{C}$ and all $z \in \mathscr{A}(a, b)$. ∎

(c) We denote the Banach algebra of all Laurent series that are absolutely convergent on $\mathscr{A}(a, b)$ and have a continuous extension to $\bar{\mathscr{A}}(a, b)$ by $\mathscr{L}_{a, b}$. Addition and multiplication are defined pointwise. The norm is the supremum norm of the function corresponding to the Laurent series. Since the limit of a uniformly convergent sequence of continuous (holomorphic) functions is again continuous (holomorphic), $\mathscr{L}_{a, b}$ is complete. The unit of $\mathscr{L}_{a, b}$ is the Laurent series with $a_0=1$ and $a_n=0$ for $n \neq 0$.

LEMMA 5.9. *The complex homomorphisms of $\mathscr{L}_{a, b}$ are point-evaluations on $\bar{\mathscr{A}}(a, b)$.*

Proof. It suffices to show that the polynomials in z and $1/z$ are dense in $\mathscr{L}_{a, b}$, since we can then copy the proof of Lemma 5.4. If $\sum_{n=-\infty}^{\infty} a_n z^n \in \mathscr{L}_{a, b}$, then $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=-\infty}^{1} a_n z^n$ can be approximated uniformly on $\bar{\mathscr{A}}(a, b)$ by polynomials in z, polynomials in $1/z$, respectively. This shows that the polynomials in z and $1/z$ are dense in $\mathscr{L}_{a, b}$. ∎

THEOREM 5.10. *Let $a, b \in \mathbb{R}$ $(a<b)$ and let $(q_n)_{n \in \mathbb{Z}}$ be a two-sided sequence of convolution type. If $\varphi(t, z) \in \operatorname{inv}(\mathscr{L}_{a, b})$ for all $t \geqslant 0$, then there exists an $h \in \mathscr{L}_{a, b}$ such that $\varphi(t, z)=e^{th(z)}$ for all $t \geqslant 0$ and all $z \in \bar{\mathscr{A}}(a, b)$. In particular, $q_n(t)=\sum_{k=0}^{\infty} g_n^{k*}(t^k/k!)$.*

Proof. First note that Theorem 5.8 implies the existence of a Laurent series h which absolutely converges on $\mathscr{A}(a, b)$ and satisfies $\varphi(t, z)=e^{th(z)}$ for all $z \in \mathscr{A}(a, b)$.

Choose c, d such that $a<c<d<b$. Consider all $0<\lambda<1$ such that $\lambda c>a$. Write $\varphi_\lambda(t, z) := \varphi(t, \lambda z)$ for these λ. It follows from Theorem 5.5 that $\varphi_\lambda(1, \cdot) \in \exp(\mathscr{W}_{c, b}) \subset \exp(\mathscr{L}_{c, b})$. Since $\varphi(1, \cdot) \in \operatorname{inv}(\mathscr{L}_{c, b})$ and $\lim_{\lambda \uparrow 1} \|\varphi_\lambda(1, \cdot)-\varphi(1, \cdot)\|=0$ in $\mathscr{L}_{c, b}$, the second statement of Theorem 1.2 implies that $\varphi(1, \cdot) \in \exp(\mathscr{L}_{c, b})$. In a similar way we see that $\varphi(1, \cdot) \in \exp(\mathscr{L}_{a, d})$. It follows from Lemma 2.1 that $\varphi(1, \cdot) \in \exp(\mathscr{L}_{a, b})$, i.e., there exists an $H \in \mathscr{L}_{a,b}$ such that $\varphi(1, z) = e^{H(z)}$ for $z \in \mathscr{A}(a, b)$. It follows from Lemma 2.1 that H and h differ by a constant. We conclude that $h \in \mathscr{L}_{a,b}$.

For the last statement, see the end of the proof of Theorem 5.5. ∎

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